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PRECISION TRAJECTORIES FOR THRUSTING VEHICLES AND THE PROPAGATION OF THE ERROR COVARIANCE MATRIX THROUGH THRUST AND COAST

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SUMMARY

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This report contains an Encke type method for a precise determination of the trajectories for thrusting vehicles. A closed form expression for the approximate trajectory is taken from the known solutions of the motion of a thrusting vehicle in a uniform gravitational field under the action of an inertially fixed thrusting vector of constant thrusting magnitude. The perturbation deviation from this known solution is integrated using conventional techniques.

In addition, the report contains an approximate solution of the variational equations for the thrusting vehicle. A method is described for obtaining an approximate solution for the propagation of the covariance matrix of the errors in the position and velocity of the vehicle due to random Gaussian variations in the initial conditions of the state, in the magnitude and direction of the thrust, and the time of the onset and termination of the thrust.

Author

AN ENCKE METHOD FOR PRECISION THRUSTING TRAJECTORIES

The equations of motion of a thrusting vehicle are given by eqs. (1) and (2)

$$\ddot{R} = -\frac{R}{r^3} + \frac{T}{m} + F \quad , \tag{1}$$

where,

$$|T| = k = -\dot{m}c \quad . \tag{2}$$

The direction of the thrust vector, T, is given by some control guidance scheme, such as that developed by Battin (Ref. 1) for the APOLLO mission. We choose an approximate representation of these equations as follows

$$\ddot{S} = -\mu \frac{S_o}{s_o^3} + \frac{T_o}{m} \tag{3}$$

where S_o, T_o are constant vectors, and

$$\dot{m} = -\frac{k}{c} \quad . \tag{4}$$

The solution of these equations is given by

$$S = S_o - \mu \frac{S_o}{s_o^3} \frac{(t - t_o)^2}{2} + \dot{S}_o(t - t_o)$$

$$+ \frac{c}{k} T_o \left(\frac{cm_o}{k}\right)\left[\frac{k(t - t_i)}{cm_o} + \left(1 - \frac{k(t - t_i)}{cm_o}\right) \log\left(1 - \frac{k(t - t_i)}{cm_o}\right)\right] \tag{5}$$

$$\dot{S} = -\mu \frac{S_o}{s_o^3}(t - t_o) + \dot{S}_o - \frac{c}{k} T_o \log\left(1 - \frac{k(t - t_i)}{cm_o}\right) \tag{6}$$

$$m = m_o - \frac{k}{c}(t - t_i) \quad . \tag{7}$$

It should be noted that provision is made in this solution for the time of thrust initiation, t_i, to be different from the initial time, t_o. This will permit the burning time to be used as an error parameter in computing the variational partial derivatives.

The Encke perturbation differentiation equations are given by

$$\ddot{P} = \ddot{R} - \ddot{S} = -\mu\left(\frac{R}{r^3} - \frac{S_o}{s_o^3}\right) + \frac{1}{m}(T - T_o) + F \quad . \tag{8}$$

The initial conditions for eq. (8) are given by

$$\begin{aligned} P(t_o) &= \dot{P}(t_o) = 0 \quad , \\ R(t_o) &= S_o \quad , \quad \dot{R}(t_o) = \dot{S}_o \quad . \end{aligned} \tag{9}$$

Integrating eq. (8), we may obtain $P(t)$ and $\dot{P}(t)$. The solutions of eqs. (1) and (2) are given by

$$\begin{aligned} R &= P + S \\ \dot{R} &= \dot{P} + \dot{S} \\ m &= m_o - \frac{k}{c}(t - t_i) \quad . \end{aligned} \tag{10}$$

In order to retain maximum accuracy in the computation of the acceleration equation, $\ddot{P}$, it is necessary to compute the difference of cubes by the following formula:

$$\frac{R}{r^3} - \frac{S_o}{s_o^3} = \frac{P + (S - S_o)}{r^3} + \frac{S_o}{s_o^3}\left(\frac{3\alpha + 3\alpha^2 + \alpha^3}{1 + (1 + \alpha)^{3/2}}\right) , \tag{11}$$

where

$$\alpha = - \frac{(P + S + S_o) \cdot [P + (S - S_o)]}{r^2} . \tag{12}$$

It would be convenient to have a similar expansion for the term $T - T_o$. Unfortunately, this is not easily derived at and it is necessary to rely on a simple machine subtraction for the computation.

Since the actual thrusting logic will require a variable thrust vector, there will arise a point where the approximating T_o is no longer a good approximation. In addition, the truncation error in integrating $\ddot{P}$ will produce numerical errors in the solution. For both reasons, it is necessary to rectify the orbit. The recommended procedure is to estimate the truncation error in P and $\dot{P}$. Whenever the truncation error affects the least significant portion of S or $\dot{S}$, a rectification should be made. The following equations serve to rectify the orbit at the time of rectification, t_r:

$$\begin{aligned} S_o &= R(t_r) \\ \dot{S}_o &= \dot{R}(t_r) \\ P(t_r) &= \dot{P}(t_r) = 0 \\ T_o &= T(t_r) . \end{aligned} \tag{13}$$

A criterion for determining the cut-off time as the required velocity to be gained (V_D) approaches zero is given below:

$$v_d = |V_D| \quad . \tag{14}$$

For small v_d, the cut-off time is given by

$$t_{co} = t + \frac{v_d}{\sqrt{\frac{\mu^2}{r^4} + \frac{k^2}{m^2} - 2\mu \frac{R \cdot T}{m r^3}}} \tag{15}$$

If $t_{co} - t$ is greater than one integration interval, we may proceed with a normal integration step. If $t_{co} - t$ is less than a normal integration step, we may integrate the equations of motion with a Runge-Kutta step for a Δt given by

$$\Delta t = t_{co} - t \tag{16}$$

SOLUTION OF THE VARIATIONAL EQUATIONS

It is often required to obtain the variation in R and $\dot{R}$ due to random errors in the initial conditions of R and $\dot{R}$ as well as errors in the thrust actuation. A procedure is presented for approximating these variations by considering the partial derivatives of eqs. (5) and (6) with respect to the variables S_o, $\dot{S}_o$, T_o, k, t_i and t_{co}.

The partial derivative matrices are listed below with their appropriate definitions.

a) State Transition Matrix

$$\frac{\partial S}{\partial s_o} = \left[1 - \frac{\mu(t - t_o)^2}{2 s_o^3}\right] I + \frac{3\mu(t - t_o)^2}{2 s_o^5} \{S_o\}[S_o] \tag{17}$$

$$\frac{\partial S}{\partial \dot{s}_o} = (t - t_o) I \tag{18}$$

$$\frac{\partial \dot{S}}{\partial s_o} = \left[- \frac{\mu(t - t_o)}{s_o^3}\right] I + \frac{3\mu(t - t_o)}{s_o^5} \{S_o\}[S_o] \tag{19}$$

$$\frac{\partial \dot{S}}{\partial \dot{s}_o} = I \tag{20}$$

The 6x6 matrix for the variation of the state with respect to the errors in the initial conditions, $\frac{\partial S(t)}{\partial s(t_o)}$, is given below:

$$\Phi = \begin{bmatrix} \frac{\partial s}{\partial s_o} & \frac{\partial s}{\partial \dot{s}_o} \\ \frac{\partial \dot{s}}{\partial s_o} & \frac{\partial \dot{s}}{\partial \dot{s}_o} \end{bmatrix} \tag{21}$$

b) <u>The Variation of the State with Respect to the Thrust Direction and Magnitude</u>

Let

$$T_o = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix}$$

$$\frac{\partial s}{\partial \ell_1} = \frac{c}{k}\left[(t - t_i) + [\frac{cm_o}{k} - (t - t_i)] \log [1 - \frac{k(t - t_i)}{cm_o}]\right] I$$

$$- \frac{c}{k^3}\left[2(t-t_i) + [\frac{2\,cm_o}{k} - (t-t_i)] \log [1 - \frac{k(t-t_i)}{cm_o}]\right] \{T_o\}^{[T_o]} \tag{22}$$

$$\frac{\partial \dot{s}}{\partial \ell_1} = - \frac{c}{k} \log [1 - \frac{k(t-t_i)}{cm_o}] I$$

$$+ \frac{c}{k^3}\left[\log [1 - \frac{k(t-t_i)}{cm_o}] + \frac{\frac{k(t-t_i)}{cm_o}}{1 - \frac{k(t-t_i)}{cm_o}}\right] \{T_o\}^{[T_o]}. \tag{23}$$

The 6×3 matrix for the variation of the state with respect to the three thrust components, $\frac{\partial S(t)}{\partial \ell}$, is given by

$$U = \begin{bmatrix} \frac{\partial S}{\partial \ell_1} \\ \\ \frac{\partial \dot{S}}{\partial \ell_1} \end{bmatrix} \tag{24}$$

c) <u>The Variation of the State with Respect to Engine Start and Cut-Off</u>

For $t_i \leq t \leq t_{co}$

$$\frac{\partial S}{\partial t_i} = \frac{c}{k} T_o \log \left[1 - \frac{k(t-t_i)}{cm_o}\right] \tag{25}$$

$$\frac{\partial \dot{S}}{\partial t_i} = - \frac{1}{m_o \left[1 - \frac{k(t-t_i)}{cm_o}\right]} T_o \tag{26}$$

$$V(t_i) = \begin{bmatrix} \frac{\partial S}{\partial t_i} \\ \\ \frac{\partial \dot{S}}{\partial t_i} \end{bmatrix} \tag{27}$$

For $t = t_{co}$

$$\frac{\partial S(t_{co})}{\partial t_{co}} = - \dot{S}(t_{co}) \tag{28}$$

$$\frac{\partial \dot{S}(t_{co})}{\partial t_{co}} = - \ddot{S}(t_{co}) \tag{29}$$

$$V(t_{co}) = \begin{bmatrix} \dfrac{\partial S(t_{co})}{\partial t_{co}} \\ \dfrac{\partial \dot{S}(t_{co})}{\partial t_{co}} \end{bmatrix} \tag{30}$$

THE PROPAGATION OF THE COVARIANCE MATRIX THROUGH THRUST

The linear variations of the position and velocity vectors may be approximated as follows:

$$\delta x = \Phi \delta x_o + U \delta \ell + V \delta t \tag{31}$$

The expected value of the covariance matrix, $E(\delta x, \delta x^*)$, is given by

$$P(t) = \Phi P(t_o) \Phi^* + U E(\delta \ell, \delta \ell^*) U^* + V E(\delta t, \delta t^*) V^* \tag{32}$$

No correlation between δx_o, $\delta \ell$, and δt is assumed to exist.

Following each rectification, Φ is returned to the 6×6 identity matrix I, t_r becomes t_i, t_o becomes t_i, $P(t_r)$ becomes the new $P(t_o)$, and T_o becomes the thrust given by the guidance law, $T(t_r)$. The entire process is repeated and the covariance matrix P is propagated through the thrust burn.

COMPUTATION OF THE THRUST ACTUATION ERROR MATRIX

The statistical model that will be used to compute the thrust actuation errors is described as follows:

1. The error in thrust magnitude, δk, will be assumed to act along the vector T.
2. The tip-off error, δx, in pointing the engine will be assumed to be symmetrically distributed about T.

With these two assumptions the actuation error covariance matrix is given by

$$E(\delta \ell, \delta \ell^*) = a_1 I + a_2(\{T_o\}^{[T_o]} - k^2 I) \tag{33}$$

where

$$\begin{aligned} a_1 &= (\delta k)^2 \\ a_2 &= (\delta k)^2 - \frac{k^2}{2} \tan^2(\delta \alpha) \end{aligned} \tag{34}$$

and I is a 3×3 unit matrix.

REFERENCES

1. Battin, R.H.; "Space Guidance Analysis," Memo No. 37, M.I.T., March 8, 1963.